Introduction to Statics

.PDF Edition – Version 0.95

Unit 11
First Moments

Helen Margaret Lester Plants
Late Professor Emerita
Wallace Starr Venable
Emeritus Associate Professor
West Virginia University, Morgantown, West Virginia

Unit 11
First Moments

Frame 11-1

*** Introduction - Prerequisites**

This program will teach you to find the first moment of a given line, area, or volume about a given axis provided you are familiar with formal integration, establishment and use of limits, and the basic concepts of analytical geometry. These are assumed pre-requisites for this unit and no attempt will be made to teach them. Should you discover difficulties in these areas you are advised to refer to your calculus book.

Go to the next frame.

* Your instructor may decide not to cover this topic if he feels that your class has not had sufficient mathematical preparation. You will be able to work all of the problems in the next unit and understand most of the discussion there even if you cannot do integration.

Correct response to preceding frame

No response

Frame 11-2

Introduction

The first moment is a mathematical quantity which occurs with sufficient frequency that it is convenient to have a name for it. It has several common uses so that it is useful to be able to compute it. This unit will teach you to compute the first moments of geometric figures with respect to specified axes.

Go to the next frame.

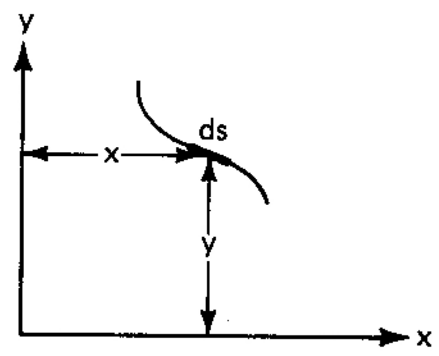
Correct response to preceding frame

No response

Frame 11-3

Definition

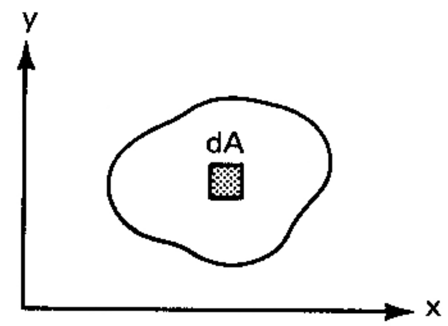
The first moment of an element with respect to a given axis is found by multiplying the element by its perpendicular distance from the axis.



1. The first moment of the element ds about the x-axis is yds.

What is the first moment of the same element about the y-axis?

____



2. What is the first moment of dA, located at (x,y) about the x-axis?

____

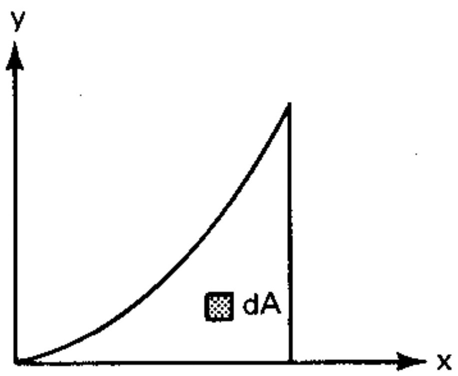
Correct response to preceding frame0

1. xds
2. ydA

Frame 11-4

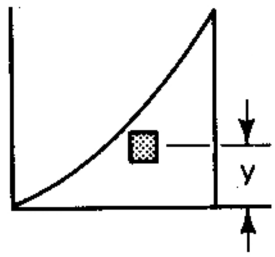
Definition

The perpendicular distance from the element to the required axis is called the moment arm.



On the figure draw the moment arm necessary to find the first moment of the element dA about the x-axis and write the expression for the required first moment.

Correct response to preceding frame



ydA

Frame 11-5

Notation

For no particularly good reason that I know, first moments are often designated by the symbol Q.

Thus, Q_x means the first moment with respect to the x-axis. For an element of area,

$$Q_x = \_\_\_\_\_ \; dA$$

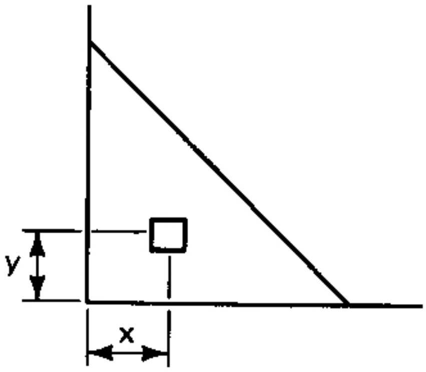
Correct response to preceding frame

$Q_x = y\,dA$

Frame 11-6

First Moments of Geometric Figures

The first moment of a body (line, area, or volume) with respect to a given axis is the sum of the first moments of all its elements with respect to that axis. This is usually written as an integral evaluated between appropriate limits.



The expression

$$Q_x = \int^A y\,dA$$

is the first moment of the triangle about the x-axis.

Write an expression for its first moment about the y-axis.

Q_y = ____________________

Correct response to preceding frame

$$Q_y = \int^A xdA$$

Frame 11-7

Transition

As you can see from what we have done so far, the idea of first moment is simple enough but the actual work-out can become messy.

The remainder of this unit is devoted to providing you with some practice in working problems at the same time that it develops the expressions for the first moments of some simple geometric areas.

Go to the next frame.

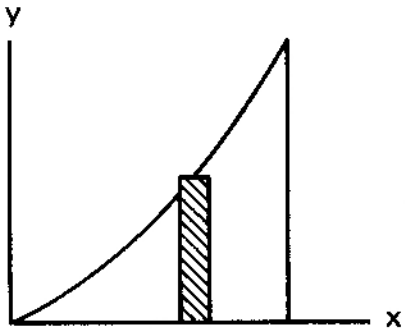
Correct response to preceding frame

No response

Frame 11-8

Selection of an Element

The computation of first moments of areas is rendered considerably less painful if care is exercised in the selection of the element of area. A good rule is to select the largest possible element parallel to the line about which you are taking moments.



In the figure above the element shown would be appropriate for taking moments about the _______ axis.

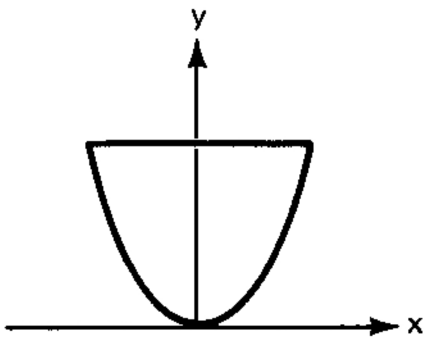
Correct response to preceding frame

The y-axis

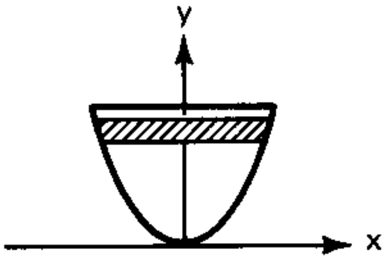
Frame 11-9

Selection of Element

On the area shown draw the largest possible element that could be used to find Q_x.



Correct response to preceding frame



The element is shown below the top since it represents a general horizontal element located at any vertical position.

Frame 11-10

Review

Complete Page 11-1 of your notebook.

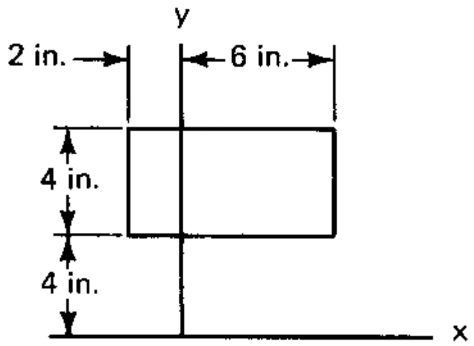
Correct response to preceding frame

No response

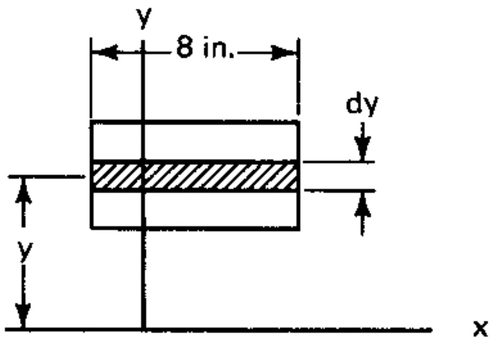
Frame 11-11

Set-up for First Moment

For the figure shown, identify the largest possible element and set up the integral, including limits, for Q_x.



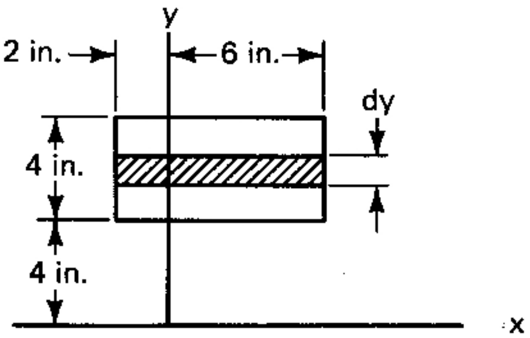
Correct response to preceding frame



$$Q_x = \int_4^8 y\ 8dy$$

Frame 11-12

Evaluation of First Moment



$$Q_x = \int_4^8 y\ 8dy$$

Evaluate Q_x and give its units.

Correct response to preceding frame

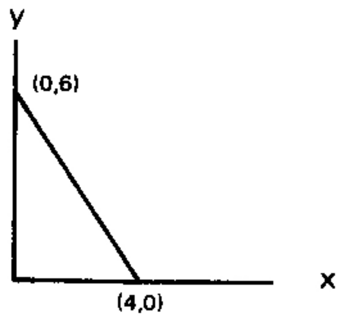
$$Q_x = \frac{8y^2}{2}\Bigg|_4^8$$

$$= 4[64 - 16]$$

$$= 192 \text{ in}^3$$

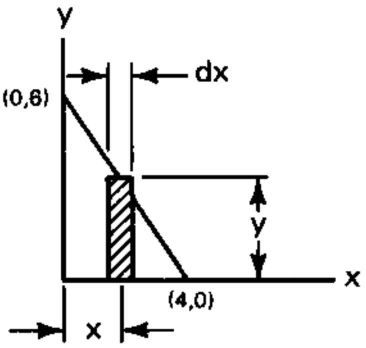
Frame 11-13

First Moment of an Area



Set up the element and the integral to find Q_y for the area shown. (Follow the steps on page 11-1 of your notebook)

Correct response to preceding frame



$$y = +6 - \frac{3}{2}x$$

$$Q_y = \int x\,dA$$

$$dA = y\,dx$$

$$Q_y = \int x\,y\,dx$$

$$Q_y = \int_0^4 x\left[6 - \frac{3}{2}x\right]dx$$

$$Q_y = \int_0^4 \left[6x - \frac{3x^2}{2}\right]dx$$

Frame 11-14

First Moment of an Area

Complete the integration and evaluate the first moment of the area in the preceding frame if the units are inches.

Correct response to preceding frame

$Q_y = 16 \text{ in}^3$

<u>Solution:</u>

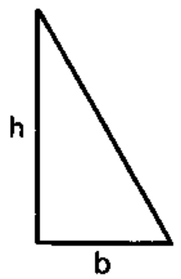
$$Q_y = \int_0^4 \left[6x - \frac{3x^2}{2}\right] dx$$

$$= \frac{6x^2}{2} - \frac{x^3}{2}\Big|_0^4$$

$$= 48 - 32 = 16$$

Frame 11-15

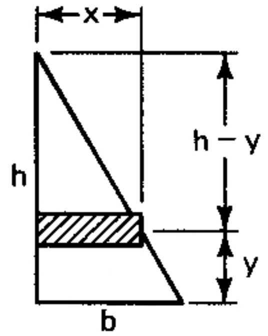
First Moment of a Triangle



Find the first moment of the triangle about base b.

Correct response to preceding frame

$$Q_b = \frac{bh^2}{6}$$



Solution:

$$Q_b = \int y\,dA$$

$$dA = x\,dy$$

$$\frac{x}{b} = \frac{h - y}{h}$$

$$dA = \frac{b}{h}(h - y)\,dy$$

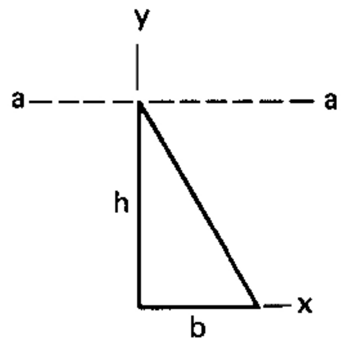
$$Q_b = \int_0^h \frac{b}{h}(h - y)y\,dy = \left.\frac{by^2}{2} - \frac{by^3}{h3}\right|_0^h$$

$$= \frac{by^2}{2} - \frac{bh^2}{3} = \frac{bh^2}{6}$$

Frame 11-16

First Moment of a Triangle

First moments may be taken about axes other than coordinate axes.

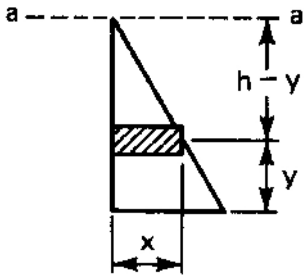


Find the first moment of the triangle about the a-a axis through its apex.

Correct response to preceding frame

$Q_{a-a} = \frac{bh^2}{3}$

Solution:



$Q_a = \int (h - y)dA$

$dA = xdy$

$\frac{x}{h - y} = \frac{b}{h}$

$dA = \frac{b}{h}(h - y)dy$

$$Q_{a-a} = \int_0^h \frac{b}{h}(h - y)^2 dy = \frac{-b}{3h}(h - y)^3 \Big|_0^h$$

$$= 0 + \frac{b}{3h}\, h^3$$

Frame 11-17

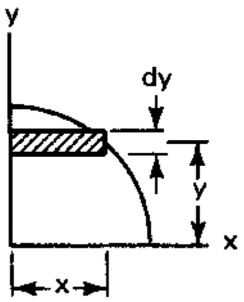
Problem

Do Problem 11-1 in your notebook.

Correct response to preceding frame

$Q_y = \frac{r^3}{3}$

Solution:



$$Q_x = \int y dA = \int_0^r yxdy$$

Equation of edge:

$$x^2 + y^2 = r^2$$

$$x = \sqrt{r^2 - y^2}$$

$$Q_x = \int_0^r y\sqrt{r^2 - y^2}\, dy = -\frac{1}{3}\sqrt{(r^2 - y^2)^3}\Big|_0^r = \frac{r^3}{3}$$

Frame 11-18

Transition

So much for the first moments of areas.

It is sometimes necessary or, at any rate, useful to be able to find the first moment of a line segment. The general method is the same but the details are a bit different as you will see in the next few frames.

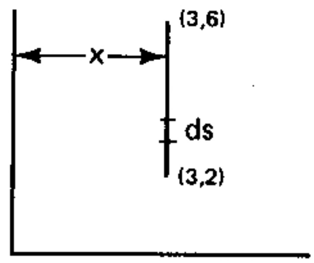
Go to the next frame.

Correct response to preceding frame

No response

Frame 11-19

First Moment of a Line



The vertical line has the equation x = 3.

The first moment about the y-axis of the line segment shown is

$$Q_y = \int x ds = \int_2^6 3dy$$

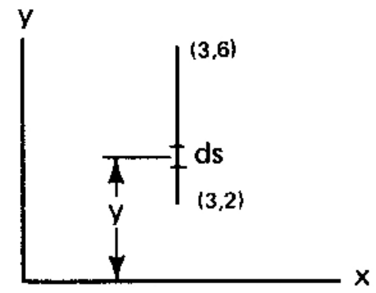
Q_y = ______ units^2

Correct response to preceding frame

$Q_y = 12 \text{ units}^2$

Frame 11-20

First Moment of a Line



Complete the following:

$Q_x = \int y \, ds$

$= \_\_\_\_\_\_\_\_\_\_\_\_$

Correct response to preceding frame

$Q_x = 16 \text{ units}^2$

Solution:

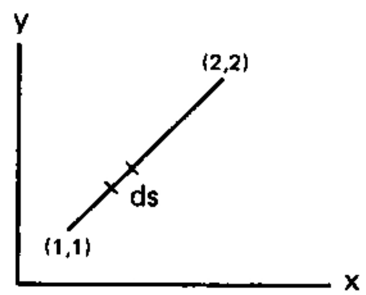
$$Q_x = \int y ds$$

$$= \int_2^6 y dy$$

$$= \frac{y^2}{2} \Big|_2^6 = \frac{36 - 4}{2}$$

Frame 11-21

First Moment of a Line



Find the first moment of the line segment shown about the y-axis remembering that

$$ds = \sqrt{dx^2 + dy^2}$$

Correct response to preceding frame

$Q_y = \frac{3}{2}\sqrt{2}$ units^2

Solution:

$$Q_y = \int x\,ds$$

line equation is

$$x = y$$

therefore $dx = dy$

$$ds^2 = dx^2 + dy^2$$

$$ds^2 = 2dx^2$$

$$ds = \sqrt{2}\,dx$$

$$Q_y = \int_1^2 \sqrt{2}\,x\,dx$$

$$= \sqrt{2}\left.\frac{x^2}{2}\right|_1^2 = \frac{3}{2}\sqrt{2}$$

Frame 11-22

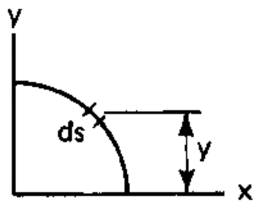
Review

Complete problem 11-2 in your notebook.

Correct response to preceding frame

$Q_x = a^2$

Solution:



$$Q_x = \int y\,ds$$

$$x^2 + y^2 = a^2$$

$$2x\,dx + 2y\,dy = 0$$

$$dx = \frac{-y\,dy}{x}$$

$$ds = \sqrt{\frac{y^2}{x^2}\,dy^2 + dy^2} = \sqrt{\frac{(y^2 + x^2)\,dy^2}{x^2}} = \sqrt{\frac{a^2}{a^2 - y^2}\,dy^2}$$

$$Q_x = \int_0^a \frac{a\,y\,dy}{\sqrt{a^2 - y^2}} = -a\sqrt{a^2 - y^2}\,\Big|_0^a = +a^2$$

Frame 11-23

Transition

Geometrical figures in two dimensions are restricted to lines and areas and we have dealt with both.

Now we move into a three dimensional world to look at the first moments of volumes about planes and lines.

Most of the material in the next section is very similar to what you have been doing in two dimensions. The work-out of problems is, however, more difficult because of the third dimension. We shall, therefore, keep the actual evaluation of integrals to a minimum and concentrate on setting them up.

Go to the next frame.

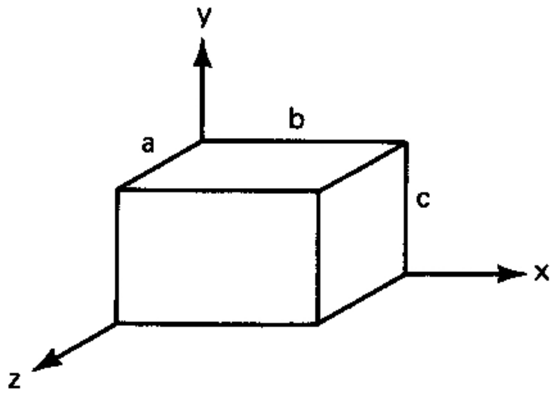
Correct response to preceding frame

No response

Frame 11-24

First Moments of Volumes

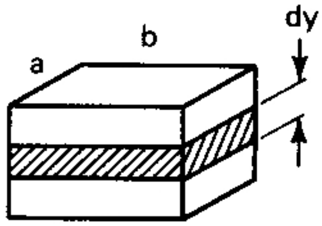
The easiest three dimensional problem is to take the first moment of a volume with respect to a given plane. To do this efficiently one chooses the largest possible element of volume which is parallel to the desired plane.



Draw, and write an expression for, the largest possible element for finding the first moment of the volume shown with respect to the xz plane.

dV = ____________________

Correct response to preceding frame



$dV = ab\ dy$

Frame 11-25

First Moment of Volumes

The next step is to multiply the element of volume by the distance from the plane. Then integrate between the proper limits.

For the volume in the preceding frame, find Q_{xz}

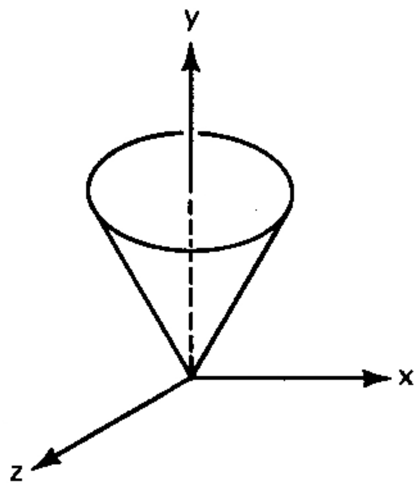
Correct response to preceding frame

$Q_{xz} = \frac{abc^2}{2}$ **Solution:**

$$Q_{xz} = \int y\,dV = \int_0^c ab\,y\,dy = \frac{aby^2}{2}\Bigg|_0^c$$

Frame 11-26

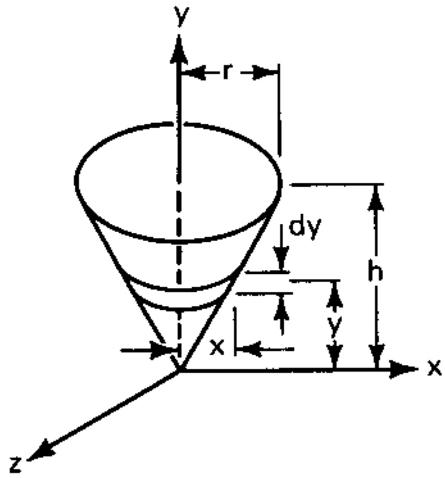
First Moments of Volumes



Draw the element and set up the integral to find Q_{xz} for a cone of radius, r , and height, h.

Correct response to preceding frame

$$Q_{xz} = \int y dV = \int_0^h y\pi x^2 dy = \int_0^h \pi y \frac{r^2}{h^2} y^2 dy$$

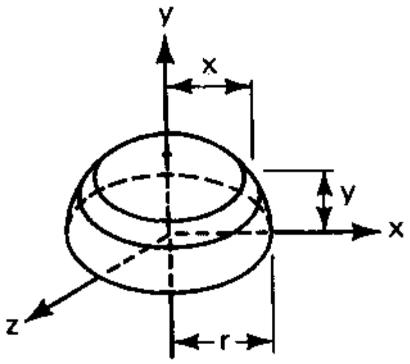


Frame 11-27

Problem

Do problem 11-3 in your notebook.

Correct response to preceding frame



$$dV = \pi x^2 dy$$

$$x^2 + y^2 = r^2$$

$$dV = \pi(r^2 - y^2)dy$$

$$Q_{xz} = \int_0^r \pi y(r^2 - y^2)dy$$

Frame 11-28

Transition

The remainder of the unit is included in the interests of completeness and possible future reference.

It will not be used in future units, nor should it be attempted by students who have not yet covered multiple integration in the mathematics sequence.

Such students may wish to skim through the material to get the gist of it but are cautioned against panic.

Go to the next frame if you want to--or if your teacher requires it. If not, skip to Frame 11-32.

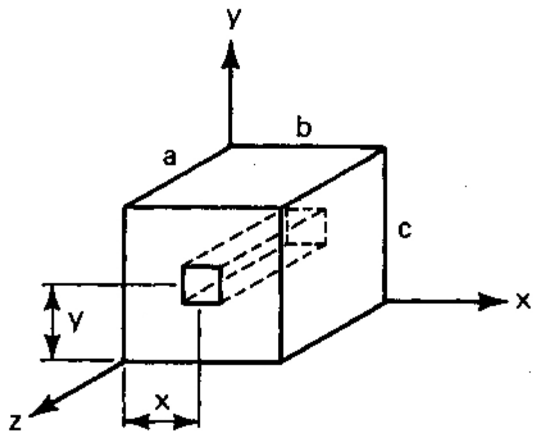
Correct response to preceding frame

No response

Frame 11-29

First Moments of a Volume About a Line

The first moment of a volume about a line becomes a little sticky. All parts of the element must be equidistant from the line in question so the element is generally a rod parallel to the line.



The proper element for Q_z is shown.

dV = ____________________________

The moment arm from the z-axis to the

element is ________________________

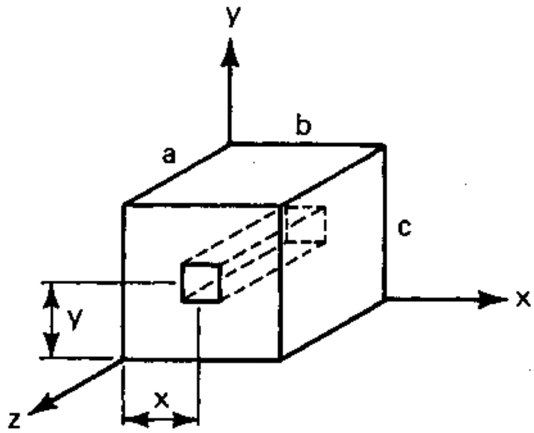
Correct response to preceding frame

$dV = a\ dx\ dy$
The moment arm is

$$\sqrt{x^2 + y^2}$$

Frame 11-30

First Moments of a Volume about a Line



The moment arm from the z-axis to the element of volume is, in terms of x and y, is

$d =$ ____________________

so the integral for the first moment is

$$Q_z = \int_{\_}^{\_} \int_{\_}^{\_} \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

Correct response to preceding frame

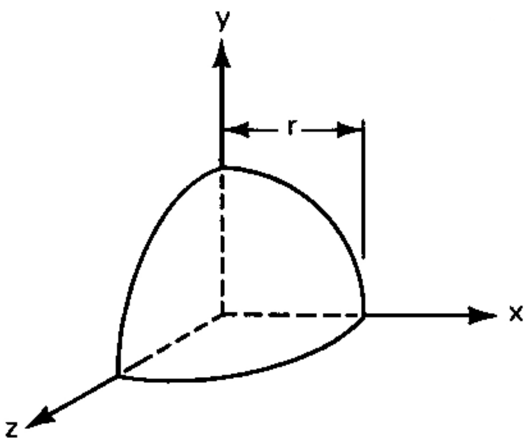
$$d = \sqrt{x^2 + y^2}$$

$$Q_z = \int_0^b \int_0^c a \sqrt{x^2 + y^2}\, dydx$$

Frame 11-31

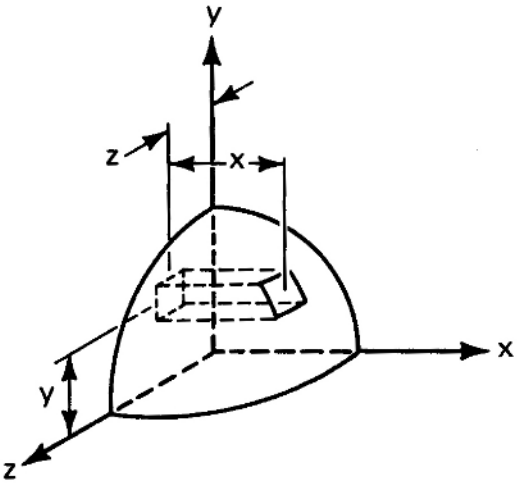
First Moments of a Volume about a Line

Draw and write an expression for the largest element of volume suitable for finding Q_x for the eighth of a sphere shown.



dV = ______________________________

Correct response to preceding frame



dV = x dy dz

Frame 11-32

First Moment of a Volume about a Line

For the volume in the preceding frame set up the integral for Q_x complete with limits.

Correct response to preceding frame

For the surface

$$x^2 + y^2 + z^2 = r^2$$

so $x = \sqrt{r^2 - y^2 - z^2}$

$$Q_x = \int_0^r \int_0^r \left[\sqrt{r^2 - y^2 - z^2}\right]\left[\sqrt{y^2 + z^2}\right] dydz$$

Note: The appropriate entries may now be made on Page 11-4 of your notebook.

Frame 11-33

Conclusion

The concept of first moments can be extended to figures as complicated as your math can handle. The principle is the same. Only the details are changed to afflict the innocent.

There are some short cuts available for some problems. You may have encountered them in your math text. You will find them there if you need them.

Don't, however, try to memorize the results of your wondrous integrations. There will be a more convenient mode of attack presented in the next unit where you will find that first moments are, alas, but means to an end.