Introduction to Statics

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Unit 28

Moments of Inertia of Geometric Areas

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Unit 28
Moments of Inertia of Geometric Areas

Frame 28-1

*** Introduction**

This unit will deal with the computation of second moments, or moments of inertia, of areas.

It will not attempt to teach you the calculus involved since you are presumed to have learned it in another course. The particular skills you will need are in establishing elements of area, writing the equations of various curves, determination of appropriate limits, and, of course, formal integration. If you feel uncomfortable with any of these it would be wise to have a calculus book handy for reference.

This topic is traditionally taught as part of statics even though it has no application to statics problems. You will probably make your first use of it in your mechanics of materials course. You will build on this material in Unit 30, Moment of Inertia of Mass, to learn concepts and techniques useful in dynamics.

The reasoning behind all this is remarkably sane. There are many more topics to be taught in mechanics of materials than in statics so that most teachers choose to put moments of inertia into the less crowded course.

Consequently you are asked to learn the units on moment of inertia to prepare for future rather than immediate needs.

It is an important and necessary tool. Just you wait and see. Go to the next frame.

* Your instructor may decide not to cover this unit if your class has not had sufficient mathematical preparation. In that case, you should be given the values needed to fill in the Properties of Areas Table in your notebook, in order to enable you to proceed into Unit 29.

Correct response to preceding frame

No response

Frame 28-2

Moment of Inertia

The expression

$$\int y^2 \, dA$$

crops up so frequently in the world of engineering that it has become convenient to have a name for it and routine methods for computing it. With great regard for economy of words we call the expression written above "the moment of inertia of the area about the x axis" or I_x for short.

Write the expression for the moment of inertia of the area about the y axis.

I_y = ______________________

Correct response to preceding frame

$I_y = \int x^2 \, dA$

Frame 28-3

First and Second Moments

The Moment of Inertia of an area is often called its "second moment". That is because the method for obtaining it is so similar to that used for finding the first moment.

For example the first moment of an area about the x-axis is given by the expression

$$Q_x = \int y \, dA$$

The moment arm, y, is raised to the first power.

If we square y (i.e., raise it to the second power) we will have the second moment of the area about the x-axis.

$$I_x = \int y^2 \, dA$$

What would you expect the expression for the third moment about the x-axis to be ?

____

Correct response to preceding frame

$\int y^3 \, dA$

I must admit that I don't know of any physical meaning for the third moment, but mathematically we can define any moment we wish.

Frame 28-4

Review

Since the determination of second moments -- moments of inertia -- is so similar to that used in finding first moments, it will be wise to take a quick look back at first moments, which we covered in Unit 11.

The symbol we used for the first moment about the y-axis is ________

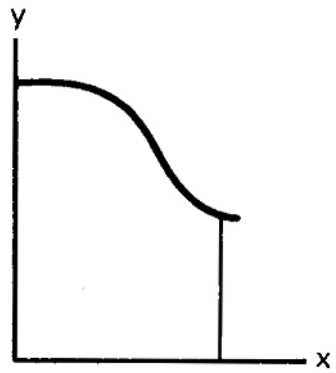
Correct response to preceding frame

Q_y

Frame 28-5

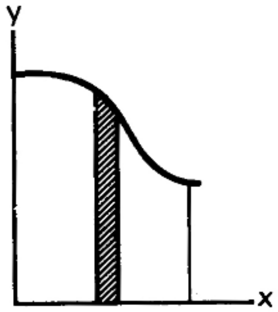
Review

On the area shown draw an appropriate dA to use in finding Q_y.



$$Q_y = \int x dA$$

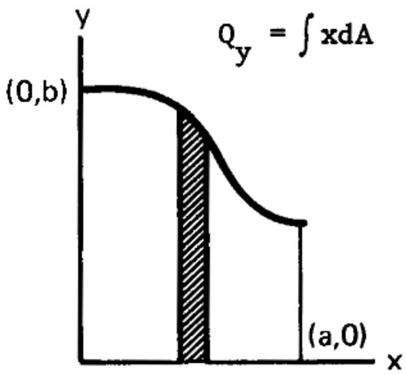
Correct response to preceding frame



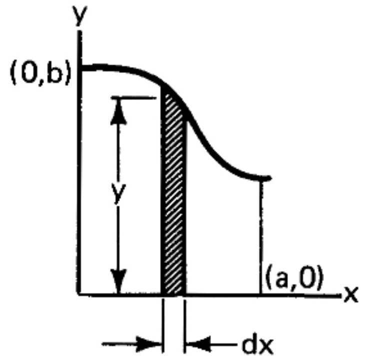
Frame 28-6

Review

Dimension the element of area in terms of x and y.



Correct response to preceding frame



Frame 28-7

Review

Using the area and element shown above write an expression for dA and determine the limits on the integration.

dA = ______________

$$Q_y = \int_{\_\_}^{\_\_} x\,dA$$

Correct response to preceding frame

$$dA = ydx$$

$$Q_y = \int_0^a xdA$$

(If this material is not coming back to you easily it would be well to review Unit 11.)

Frame 28-8

Review

You now have the expression

$$Q_y = \int x\,dA = \int_0^a x\,ydx$$

The last step before integration would be to write _______ as a function of _______ .

Correct response to preceding frame

y as a function of x

Frame 28-9

Transition

So much for review.

Now we are ready to begin on second moments. Only the vocabulary and basic expression have been changed to confuse the innocent. The basic attack is the same.

Go to the next frame.

Correct response to preceding frame

No response

Frame 28-10

Equation

The second moment of an area about the y-axis is given by the expression

$I_y =$ ______________________________

Correct response to preceding frame

$$I_y = \int x^2 \, dA$$

Frame 28-11

Selection of an Element

The first step in finding second moments is the same as the first step in finding first moments. You must choose an appropriate element of area using the same criterion you used for the selection of dA when finding the first moment of area. The rule you used then was "take the largest possible element which is all the same distance from

the ________________________________ ."

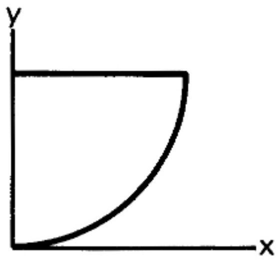
It usually turned out to be a slender bar parallel to the _______________________ .

Correct response to preceding frame

reference axis
reference axis

Frame 28-12

Equation and Element

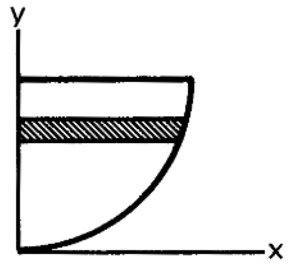


1. Write the expression for the moment of inertia about the x-axis for the area shown.

2. Draw on the figure an appropriate dA for your equation.

Correct response to preceding frame

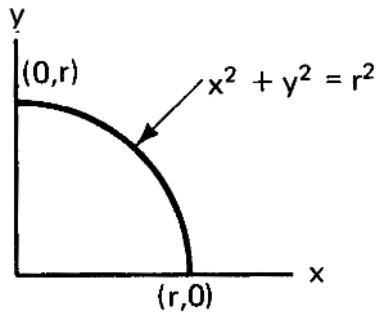
$$I_x = \int y^2 dA$$



Frame 28-13

Selection of an Element

For the quarter circle shown



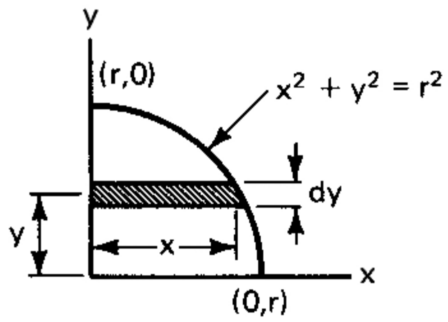
$$I_x = \int y^2 dA$$

Draw the appropriate dA on the figure and dimension it.

Write the expression for dA.

dA = ______________________

Correct response to preceding frame



$$dA = x\,dy$$

Frame 28-14

Moment of Inertia

For the figure in the preceding frame

1. Set up the integral

$$I_x = \int y^2 dA$$

all in terms of y.

2. Determine the limits.

Correct response to preceding frame

$$I_x = \int_0^r y^2 \sqrt{r^2 - y^2}\, dy$$

<u>Solution:</u>

$$I_x = \int y^2 dA$$

$$= \int_0^r y^2 x dy$$

$$x^2 + y^2 = r^2$$

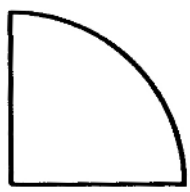
$$x = \sqrt{r^2 - y^2}$$

$$I_x = \int_0^r y^2 \sqrt{r^2 - y^2}\, dy$$

Frame 28-15

Moment of Inertia

Integrate the expression in the preceding frame and evaluate your result between the appropriate limits, to find the moment of inertia of a quarter circle about a horizontal edge.



I_x = ____________

Correct response to preceding frame

$I_x = \frac{1}{16} \pi r^4$

<u>**Solution:**</u>

(From Table of Integrals)

$$I_x = \int_0^r y^2 \sqrt{r^2 - y^2}\, dy$$

$$= \left[\frac{y}{8} \left(2y^2 - r^2 \right) \sqrt{r^2 - y^2} + \frac{r^4}{8} \text{arc sin } \frac{y}{r} \right] \Bigg|_0^r$$

$$= \frac{r}{8} \left(2r^2 - r^2 \right) \sqrt{0} + \frac{r^4}{8} \text{arc sin } (1)$$

$$- \frac{0}{8} \left(- r^2 \right) \sqrt{r^2} - \frac{r^4}{8} \text{arc sin } (0)$$

$$= 0 + \frac{r^4}{8} \left(\frac{\pi}{2} \right) - 0 - 0$$

Frame 28-16

Finding Moment of Inertia

The steps in finding the moment of inertia of an area are:

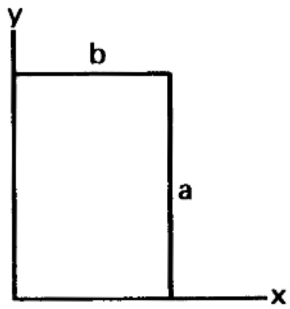
1. Draw an appropriate element.
2. Form an expression for dA.
3. ____
4. Select the limits.
5. Integrate and evaluate.

Correct response to preceding frame

3. Write the expressions for I in terms of one variable. (Or equivalent response)

Frame 28-17

Moment of Inertia

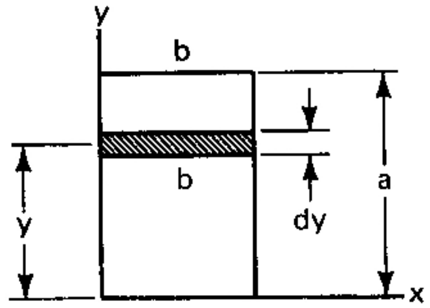


Find the moment of inertia of the rectangle shown about the x-axis.

Correct response to preceding frame

$I_x = \frac{ba^3}{3}$

Solution:



$$I_x = \int y^2 dA$$

$$dA = b\,dy$$

$$I_x = \int_0^a y^2 b\ dy = \frac{ba^3}{3}$$

Frame 28-18

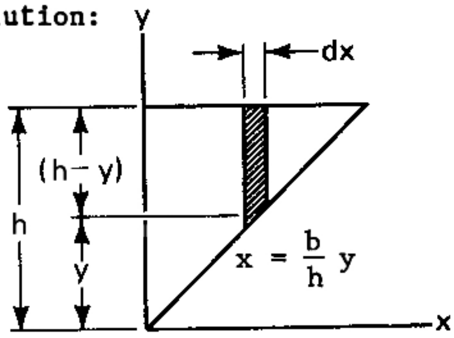
Notebook

Complete Page 28-1 of your notebook.

Correct response to preceding frame

$$I_y = \frac{hb^3}{12}$$

Solution:



$$dA = (a-y)\ dx$$

$$I_y = \int x^2 dA$$

$$= \int_0^b x^2 \left(h - \frac{h}{b} x\right) dx = \int_0^b \left(hx^2 - \frac{h}{b} x^3\right) dx$$

$$= \left(\frac{hx^3}{3} - \frac{h}{b}\frac{x^4}{4}\right)\Bigg|_0^b$$

Frame 28-19

Transition

As you can see, the possibilities for rather nasty integrations are infinite. Any calculus book worth its salt can furnish you with any number of horrible examples.

For simplicity we began by finding the moment of inertia of figures about axes along their edges. Actually the most used axes are those passing through the centroids of areas. Consequently we shall devote the next group of frames to the determination of centroidal moments of inertia.

These are not hard but they represent the basic building blocks for a great many important problems.

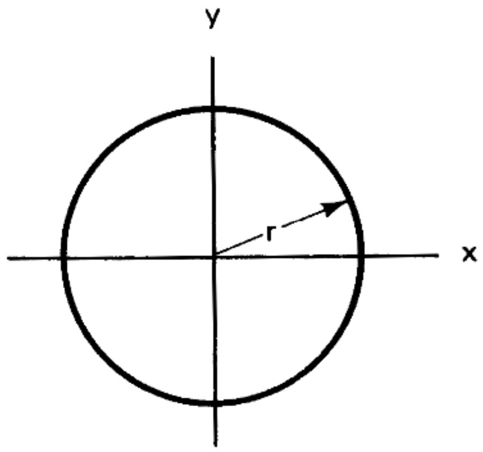
Go to the next frame.

Correct response to preceding frame

No response

Frame 28-20

Moment of Inertia

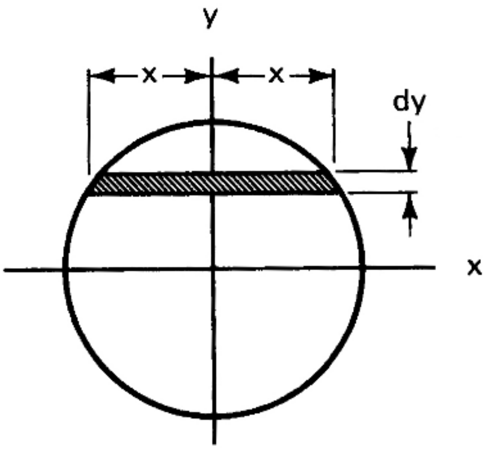


Determine an appropriate element to use in finding I_x for the circle.

Draw it on the figure and dimension it.

dA = ___________________________

Correct response to preceding frame



dA = 2xdy

Frame 28-21

Moment of Inertia

Set up the integral for I_x for the figure shown above. Determine the limits and express everything in terms of one variable.

Correct response to preceding frame

$$I_x = \int y^2 \; dA$$

$$dA = 2x\,dy$$

$$x^2 + y^2 = r^2$$

$$x = \sqrt{r^2 - y^2}$$

$$I_x = \int_{-r}^{r} 2\,y^2 \sqrt{r^2 - y^2} \; dy$$

Frame 28-22

Moment of Inertia of a Circle

Evaluate the integral from the preceding frame.

The moment of inertia of a circle about any diameter is _______________ .

Is a diameter always a centroidal axis? ☐ Yes ☐ No

Correct response to preceding frame

$I_x = \frac{\pi r^4}{4}$

$I_D = \frac{\pi r^4}{4}$

Yes

Solution:

(From Table of Integrals)

$$I_x = \int_{-r}^{r} 2y^2 \sqrt{r^2 - y^2}\, dy$$

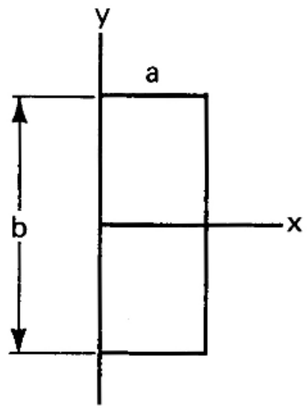
$$= 2 \left\{ \frac{y}{8}\left(2y - r^2\right)\sqrt{r^2 - y^2} + \frac{r^4}{8} \text{ arc sin } \frac{y}{r} \right\} \Bigg|_{-r}^{r}$$

$$= 2 \left[0 + \frac{r^4}{8} \text{ arc sin } (1) - 0 - \frac{r^4}{8} \text{ arc sin } (-1) \right]$$

$$= 2 \frac{r^4}{8} \left(\frac{\pi}{2} - \left(- \frac{\pi}{2} \right) \right) = \frac{\pi r^4}{4}$$

Frame 28-23

Moment of Inertia of a Rectangle



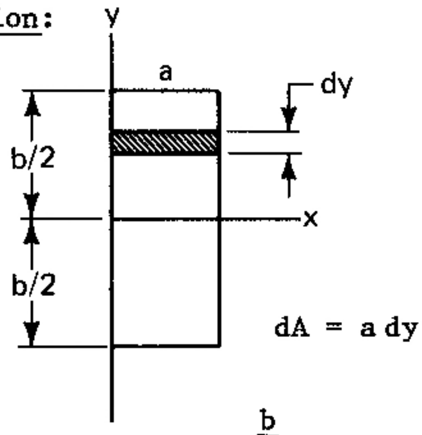
The x-axis passes through the centroid of the rectangle.

Find I_x = ______________________

Correct response to preceding frame

$I_x = \frac{ab^3}{12}$

Solution:



$$I_x = \int y^2 \, dA = \int_{-\frac{b}{2}}^{\frac{b}{2}} y^2 a \, dy$$

$$= \frac{ay^3}{3} \Bigg|_{-\frac{b}{2}}^{\frac{b}{2}}$$

$$= \frac{a}{3}\left[\frac{b}{8}^3 - \left(-\frac{b}{8}^3\right)\right]$$

$$= \frac{2ab^3}{24}$$

Frame 28-24

Notation

Hereafter a centroidal axis will be denoted by the subscript "G." Thus moment of inertia about a horizontal axis through the centroid will be written I_{xG}.

What is the symbol for the second moment about a vertical centroidal axis? _______

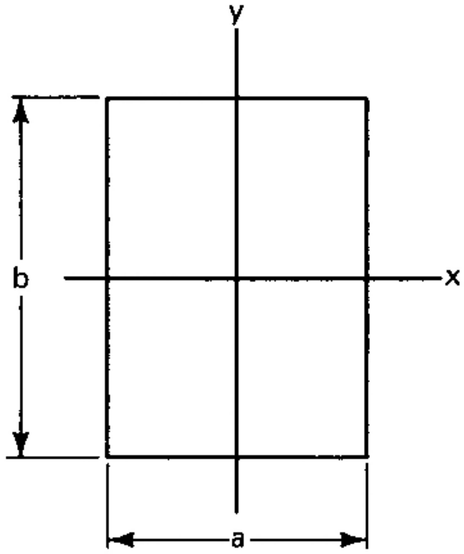
Correct response to preceding frame

I_{yG}

Frame 28-25

Rectangle

For the rectangle shown



For the rectangle shown

$$I_{xG} = \frac{ab^3}{12}$$

$$I_{yG} = \underline{\qquad\qquad}$$

Correct response to preceding frame

$$\frac{ba^3}{12}$$

Frame 28-26

Rectangle

The general expression for the moment of inertia of a rectangle about a centroidal axis parallel to one side is

$$I_G = \frac{bh^3}{12}$$

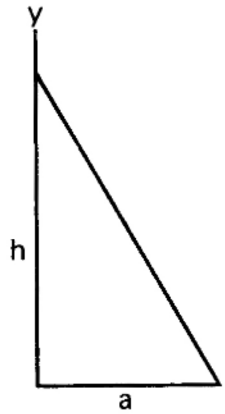
Where _____ is the dimension parallel to the axis, and _____ is the dimension perpendicular to the axis.

Correct response to preceding frame

b is parallel to the axis.
h is perpendicular to the axis.

Frame 28-27

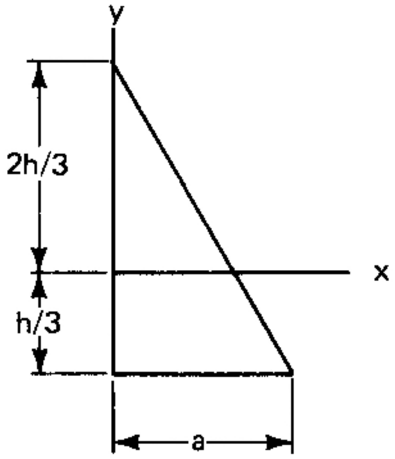
Right Triangle



1. On the triangle shown draw an x-axis through the centroid of the triangle.

2. Write the equation of the hypotenuse.

Correct response to preceding frame



$$y = mx + b$$

$$m = \text{slope} = -\frac{h}{a}$$

$$b = y \text{ intercept} = \frac{2h}{3}$$

$$y = -\frac{h}{a}x + \frac{2}{3}h$$

Frame 28-28

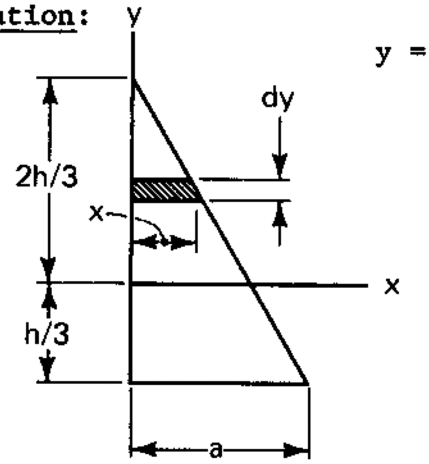
Right Triangle

Complete Problem 28-2 in your notebook.

Correct response to preceding frame

$$I_x = \frac{ah^3}{36}$$

Solution:



$$y = \frac{-hx}{a} + \frac{2}{3}h$$

$$dA = xdy$$

$$I_x = \int y^2 \, dA = \int y^2 \, x\,dy$$

$$y = -\frac{hx}{a} + \frac{2}{3}h$$

So $x = \frac{2}{3}a - \frac{a}{h}y$

$$I_x = \int_{-\frac{h}{3}}^{\frac{2h}{3}} y^2 \left(\frac{2}{3}a - \frac{a}{h}y\right) dy$$

$$= \frac{ah^3}{36}$$

Frame 28-29

Right Triangle

The general expression for the moment of inertia of a right triangle about a centroidal axis parallel to a side is

$$I_G = \frac{bh^3}{36}$$

where _____ is the dimension perpendicular to the axis.

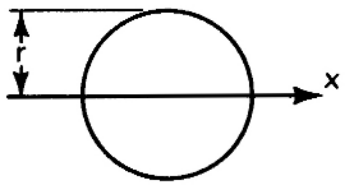
Correct response to preceding frame

h

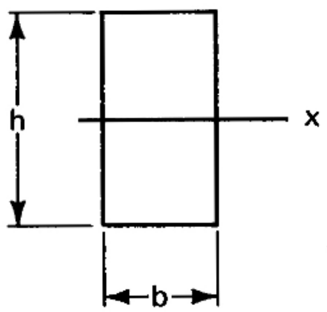
Frame 28-30

Centroidal Moments of Inertia

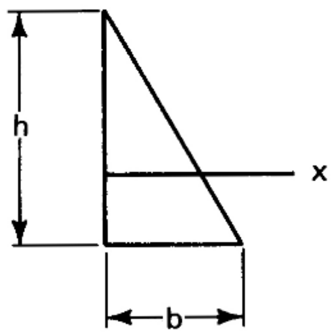
You will find it advantageous to know (all right, memorize, at least temporarily) the moments of inertia of a few geometric shapes. See how many of these you can remember.



I_{xG} = __________

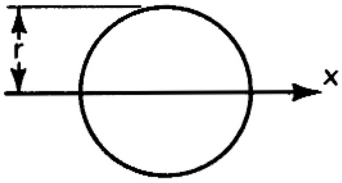


I_{xG} = __________

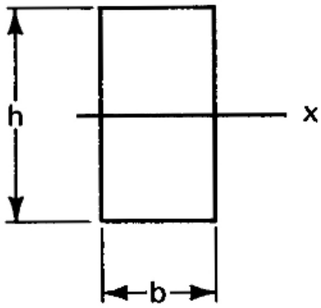


I_{xG} = __________

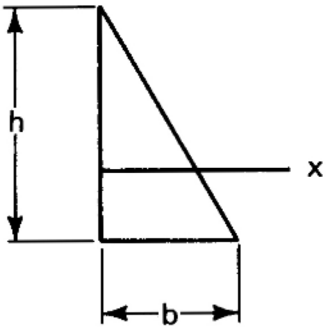
Correct response to preceding frame



$$I_{xG} = \frac{\pi r^4}{4}$$



$$I_{xG} = \frac{bh^3}{12}$$



$$I_{xG} = \frac{bh^3}{36}$$

Don’t worry if you missed some. You will be using these a great deal from now on and will learn them almost automatically.

Frame 28-31

Transition

You have now had considerable practice in setting up second moment problems, and should feel fairly confident of the method of attack. The basic set-up is always the same but the mathematical details can get sticky.

Math books will furnish you both with sticky problems and with some shortcuts to their solution. These matters are, however, beyond the scope of this unit.

The remaining frames will deal with two topics related to the second moments you have been studying. These are polar moments of inertia and radius of gyration.

Neither is hard and you should be finished in about twenty minutes.

Go to the next frame.

Correct response to preceding frame

No response

Frame 28-32

Polar Moment of Inertia

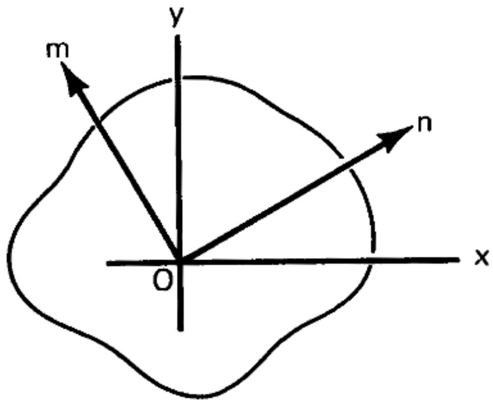
Read the section on polar moments on Page 28-3 in your notebook.

Correct response to preceding frame

No response

Frame 28-33

Polar Moment of Inertia



1. $J_o = I_x +$ ________

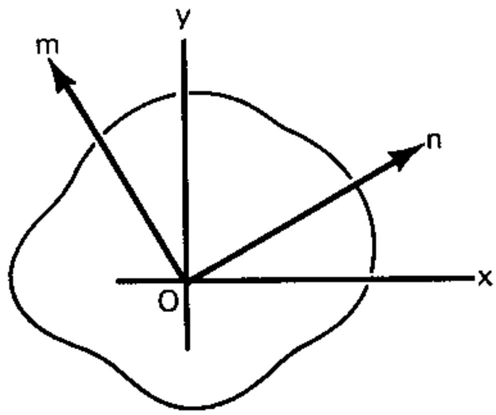
2. $J_o = I_m +$ ________

Correct response to preceding frame

1. $J_0 = I_x + I_y$
2. $J_0 = I_m + I_n$

Frame 28-34

Polar Moment of Inertia



Decide whether the following statements are true or false.

1. $I_x + I_y = I_m + I_n$

☐ True ☐ False

2. The sum of the moments of inertia of an area about any two perpendicular axes is a constant.

☐ True ☐ False

Correct response to preceding frame

Both statements are true.

Frame 28-35

Polar Moment of Inertia

Write an equation for the polar moment of inertia of an area as an integral.

J_0 = ____________________

Write the same expression as a sum.

J_0 = ____________________

Correct response to preceding frame

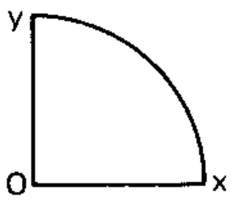
$$J_o = \int \rho^2 \, dA$$

$$J_o = I_x + I_y \qquad \text{or} \qquad \int x^2 \, dA + \int y^2 \, dA$$

Frame 28-36

Polar Moment of Inertia

For the quarter circle



$$I_x = \frac{\pi r^4}{16}$$

$I_y =$ ______________

$J_o =$ ______________

Correct response to preceding frame

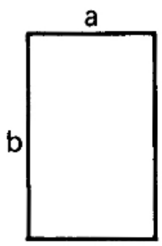
$$I_y = \frac{\pi r^4}{16}$$

$$J_o = I_x + I_y$$

$$= \frac{2\pi\ r^4}{16} = \frac{\pi r^4}{8}$$

Frame 28-37

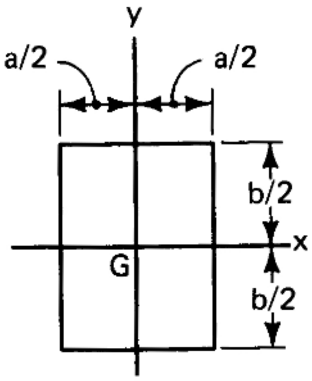
Polar Moment of Inertia



For the rectangle shown

1. Locate the centroid on the sketch.
2. Write I_{xG} = ______________________________
3. Write I_{xG} = ______________________________
4. Write J_G = ______________________________

Correct response to preceding frame



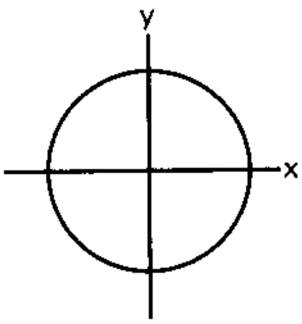
$$I_{xG} = \frac{ab^3}{12}$$

$$I_{yG} = \frac{ba^3}{12}$$

$$J_G = \frac{ab}{12}(a^2 + b^2)$$

Frame 28-38

Polar Moment of Inertia



Find J_G for the circle.

Correct response to preceding frame

$$J_G = \frac{\pi r^4}{2}$$

Solution:

$$J_G = I_{xG} + I_{yG} = \frac{\pi r^4}{4} + \frac{\pi r^4}{4}$$

Frame 28-39

Radius of Gyration

Read the material on radius of gyration in your notebook.

Correct response to preceding frame

No response

Frame 28-40

Radius of Gyration

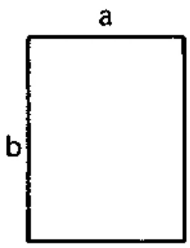
Write the equation for the radius of gyration of an area about the y-axis.

Correct response to preceding frame

$$k_y = \sqrt{\frac{I_y}{A}}$$

Frame 28-41

Radius of Gyration



Find the radius of gyration of the area shown about a vertical axis through the centroid.

Correct response to preceding frame

$k_y = \frac{a}{\sqrt{12}}$

Solution:

$$k_y = \sqrt{\frac{I_y}{A}}$$

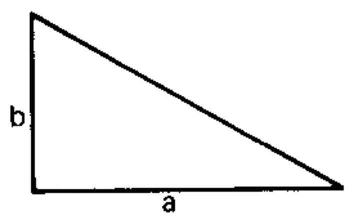
$$I_y = \frac{ba^3}{12}$$

$$A = ba$$

$$k_y = \sqrt{\frac{ba^3}{12ab}} = \sqrt{\frac{a^2}{12}}$$

Frame 28-42

Radius of Gyration



Find the radius of gyration of the triangle about a horizontal axis through the centroid

Correct response to preceding frame

$$k_{xG} = \frac{b}{3\sqrt{2}}$$

Solution:

$$I_{xG} = \frac{ab^3}{36}$$

$$A = \frac{ab}{2}$$

$$k_{xG} = \sqrt{\frac{b^2}{18}}$$

Frame 28-43

Radius of Gyration

Every moment of inertia may be written as Ak^2. Some tables give the moment of inertia in this form.

For a triangle about a centroidal axis

$$A = \frac{bh}{2} \qquad k_x = \frac{b}{3\sqrt{2}}$$

so that

$$I_x = A\,k_x^2 = \left(\frac{bh}{2}\right)\left(\frac{b}{3\sqrt{2}}\right)^2 = \frac{bh^3}{36}$$

Write the moment of inertia of a rectangle about a centroidal axis as the product of the area and the radius of gyration squared.

Correct response to preceding frame

$$I_{xG} = A\left(\frac{h}{2\sqrt{3}}\right)^2$$

<u>Solution:</u>

$$I_{xG} = \frac{bh^3}{12} \qquad A = bh$$

$$I_{xG} = (bh)\,\frac{h^2}{12}$$

$$k_{xG}^2 = \frac{h^2}{12} = \left(\frac{h}{2\sqrt{3}}\right)^2$$

Frame 28-44

Notebook

Complete the entries for the triangle, the rectangle, and the full circle in the Properties of Areas Table in your notebook. (The fourth entry will be dealt with in the next unit.) Then return to the next frame.

Correct response to preceding frame

No response

Frame 28-45

Summary

This unit has given you practice in the basic computation of moment of inertia, polar moment of inertia, and radius of gyration for areas.

Moment of inertia about an axis is necessary in the computation of bending stress; polar moment of inertia is used to find twisting stress; and radius of gyration is used in computing column loads.

There is simply no way around learning to handle these properties. You may not like them but you will certainly use them.