# Introduction to Statics

.PDF Edition – Version 0.95

## Unit 22
# Equivalent Force Systems

**Helen Margaret Lester Plants**
Late Professor Emerita
**Wallace Starr Venable**
Emeritus Associate Professor
**West Virginia University, Morgantown, West Virginia**

# Unit 22
## Equivalent Force Systems

Frame 22-1

*** Introduction**

In earlier units you have been introduced to the concept of resultants. You have already learned to find the resultants of concurrent force systems and of coplanar force systems.

In this unit you will learn to find resultants of force systems which include couples and of non-coplanar, non-concurrent force systems.

Perhaps before you begin you will want to glance back at the work you have already done on resultants. (You may find that Unit 10 is particularly useful.)

When you are ready, go to the next frame.

*This topic is sometimes excluded from a short statics course. Check your schedule to see if your instructor requires you to study it at this time.

Correct response to preceding frame

No response

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Frame 22-2

**Review**

The resultant of a concurrent force system is a (*force, couple*), given by the expression

$\overline{R} =$ _____

The line of action of the resultant of a concurrent force system always passes through the point _____ .

Correct response to preceding frame

force
$\overline{R} = \Sigma \overline{F}$
the *point of concurrency* or the point where the forces meet

(Or equivalent response)

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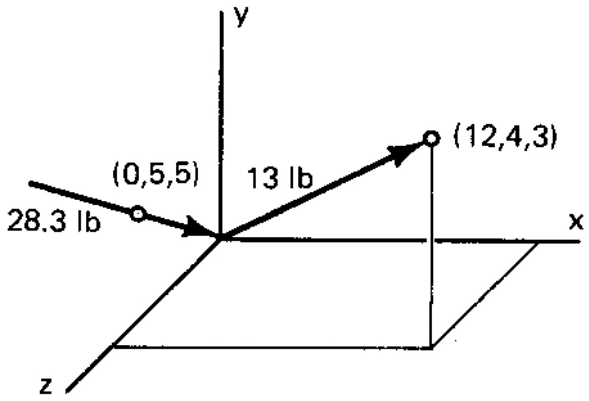
Frame 22-3

**Review**

Find the resultant of the force system shown.



Correct response to preceding frame

$\bar{R} = 12\bar{i} - 16\bar{j} - 17\bar{k}\ \overline{lb}$

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Frame 22-4

**Review**

In the problems you have seen so far the resultant of a coplanar force system has always been a (*force, couple*) given by the expression

$\bar{R} =$ _____

In these problems the location of the resultant from some point A has been obtained by using the expression

$\bar{a}_A \times \bar{R} =$ _____

Correct response to preceding frame

force
$\overline{R} = \Sigma \overline{F}$
$\overline{a}_A \times \overline{R} = \Sigma \overline{M}_A$

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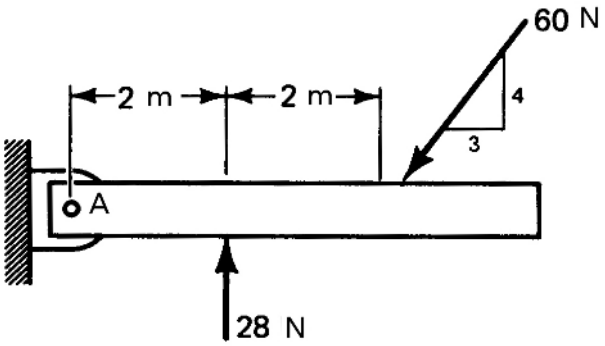
Frame 22-5

## Review

Find the resultant of the force system shown and determine where its line of action intersects the bar.



Correct response to preceding frame

$\bar{R} = -36\bar{i} - 20\bar{j}$ N
the line of action of $\bar{R}$ intersects the bar 6.8 m to the right of A

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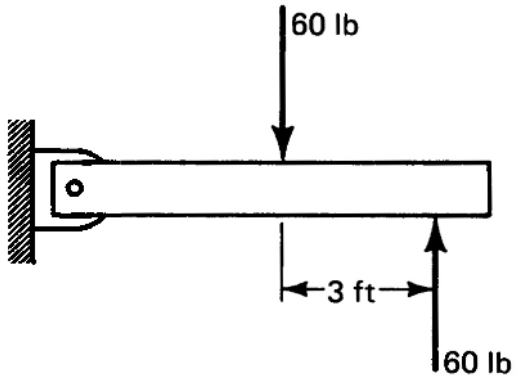
Frame 22-6

**Resultants of Coplanar Systems**




1. Is the system shown coplanar? ☐ Yes ☐ No

2. Is its resultant a force? ☐ Yes ☐ No

3. Is the resultant of a coplanar system always a force? ☐ Yes ☐ No

4. If your answer was "yes," find the resultant of the system above. If your answer was "no," tell what else the resultant could be.

Correct response to preceding frame

1. Yes
2. No
3. No
4. A couple

---

Frame 22-7

## Resultant of a Coplanar Force System

It is possible for the resultant of a coplanar force system to be a force or a couple.

1. If $\sum \overline{F} = 0$
   $\sum \overline{M}_0 \neq 0$
   the resultant is a (*force, couple*)

2. If $\sum \overline{F} \neq 0$
   $\sum \overline{M}_0 \neq 0$
   the resultant is a (*force, couple*)

3. If $\sum \overline{F} \neq 0$
   $\sum \overline{M}_0 = 0$
   the resultant is a (*force, couple*)

Correct response to preceding frame

1. resultant is a *couple*
2. resultant is a *force*
3. resultant is a *force* (and passes through 0)

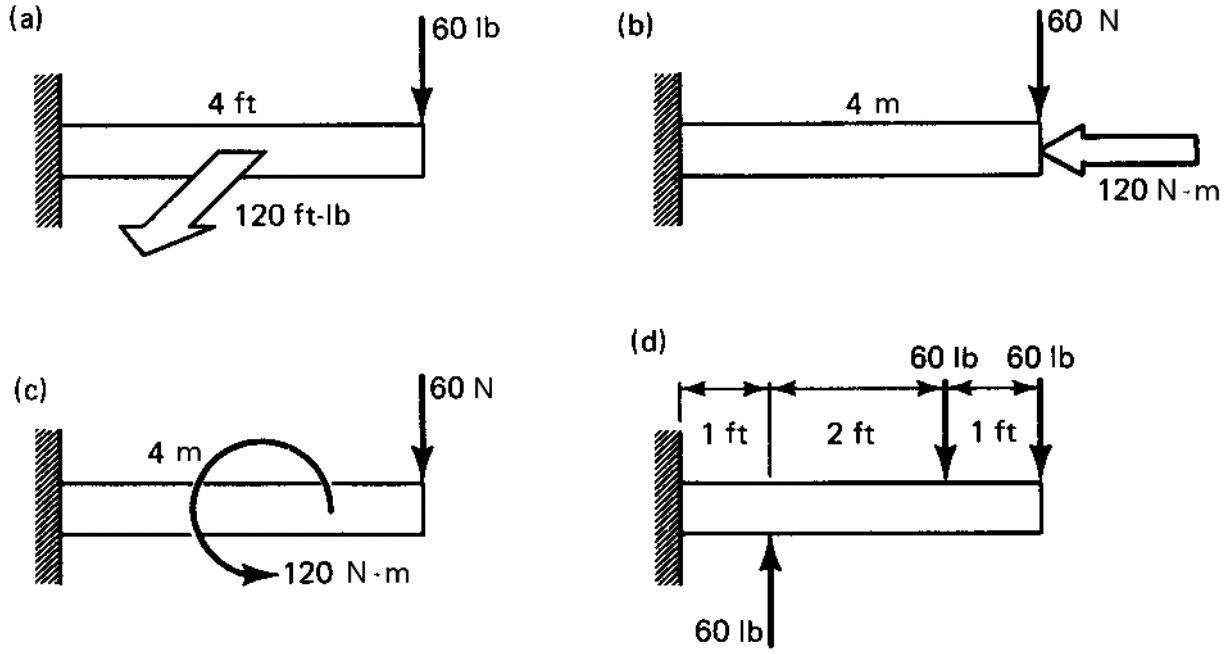
---

Frame 22-8

**Coplanar Force System**

A force system made up entirely of forces acting in a plane and/or couples normal to the plane is coplanar.

Which of the systems shown is coplanar?



Correct response to preceding frame

All except (b) are coplanar. Both (a) and (c) could easily have been represented as a pair of equal and opposite forces in the plane of the paper.

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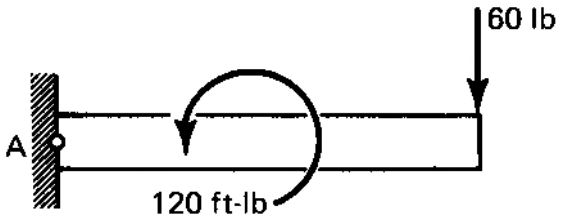
Frame 22-9

**Resultant of a Coplanar Force System**




The system shown above can be replaced by a single force. How would you have to locate that force with respect to A to get the same moment about A?

- ☐ Nearer to A than the given force of 60 pounds.
- ☐ Farther from A than the given force of 60 pounds.

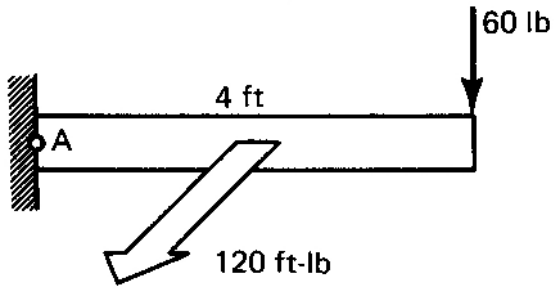
Correct response to preceding frame

Nearer to A

---

Frame 22-10

**Resultant of Coplanar Systems**




Find the resultant of the system shown and determine the horizontal distance from A to its line of action.

$\overline{R} = \Sigma\overline{F} =$ _____

$\overline{a}_A \times \overline{R} = \Sigma\overline{M}_A =$ _____

$x_A =$ _____

Correct response to preceding frame

$\bar{R} = -60\bar{j}$ lbs

$x_A = 2$ ft

**Solution:**

$\bar{a}_A \times \bar{R} = (x_A\bar{i}) \times (-60\bar{j}) = -60x_A\bar{k}$

$\bar{M}_A = 120\bar{k} - 240\bar{k} = -120\bar{k}$

$-60x_A = -120$

---

Frame 22-11

**Resultant of a Coplanar Force System**

You have seen that the resultant of a coplanar force system can be either a force or a couple. Can it also be a force and a couple?

☐ Yes ☐ No

Why or why not? _____

_____

Correct response to preceding frame

No
A coplanar system composed of a force and couple can always be further reduced to a single force in a different location. (For example, refer to Frame 22-8)

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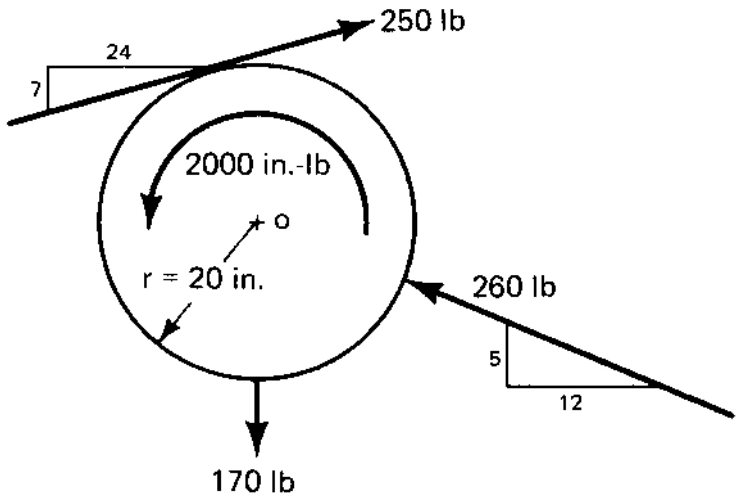
Frame 22-12

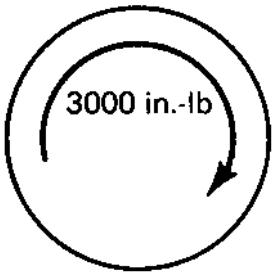
**Resultant of a Coplanar Force System**




Find the resultant of the force system shown and show it on the sketch.

Correct response to preceding frame

$\Sigma \bar{F} = 0$
$\Sigma \bar{M}_0 = -3000\bar{k}$ in-lb
Therefore the resultant is a couple.




(Or equivalent response)

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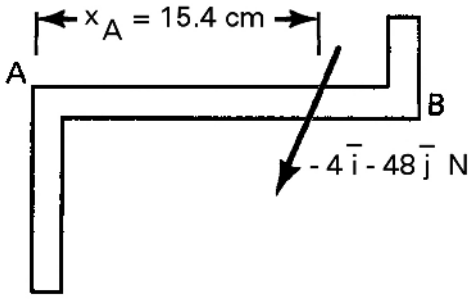
Frame 22-13
**Resultant of a Coplanar Force System**

Do Problem 22-1 in your notebook.

Correct response to preceding frame




**Solution:**

$$\bar{R} = \sum \bar{F} = -40\bar{i} + 36\bar{i} - 48\bar{j} = -4\bar{i} - 48\bar{j} \text{ N}$$

$$\bar{a}_A \times \bar{R} = \sum \bar{M}_A$$

$$(x_A\bar{i}) \times [-4\bar{i} - 48\bar{j}] = 6\bar{k} + [0.2\bar{i} + 0.1\bar{j}] \times [36\bar{i} - 48\bar{j}]$$

$$-48x_A\bar{k} = 6\bar{k} - 9.6\bar{k} - 3.6\bar{k}$$

$$-48x_A = -7.4$$

$$x_A = 0.154 \text{ m or } 15.4 \text{ cm}$$

---

Frame 22-14

**Transition**

In the preceding frames you have been finding the resultants of force systems which either result in a couple or include a couple.

The next section of the unit will teach you to find the resultant of a system of couples.

The next transition is only six frames away and is a better place than this to take a break. Continue if it is at all possible.

Go to the next frame.

Correct response to preceding frame

No response

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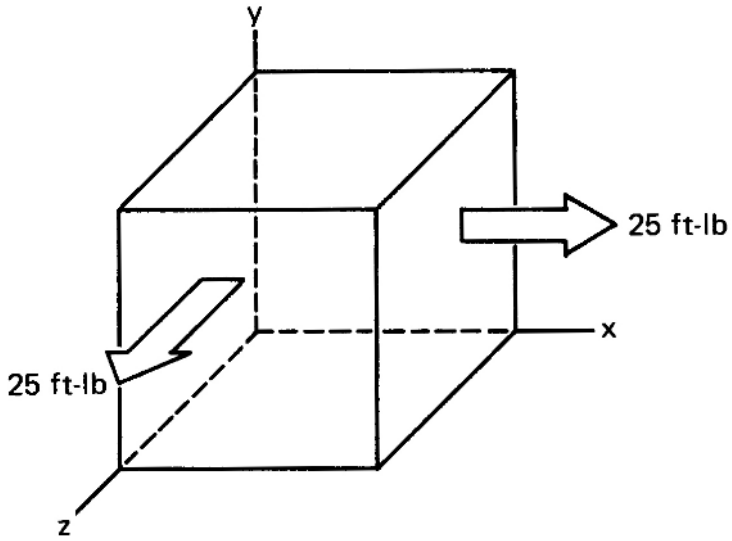
Frame 22-15

## Resultant of Couples

Since couples are vector quantities they may be added by vector addition.

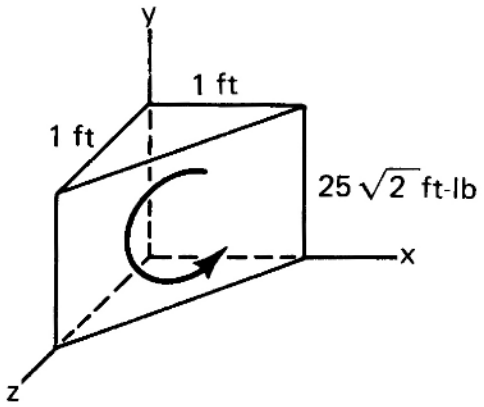



What is the resultant of the two couples shown?

Show it on a sketch.

Correct response to preceding frame

$\overline{R} = 25\overline{i} + 25\overline{k}$



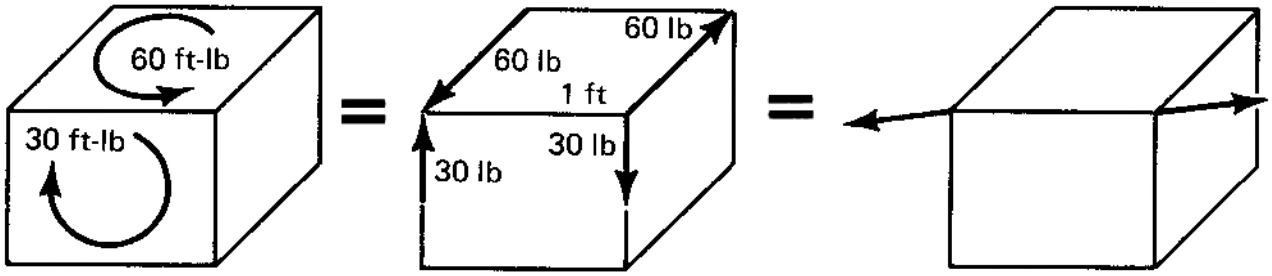

This is one way to show it. There are others.

---

Frame 22-16

**Resultant of Couples**

One way that couples are often added is by using a two force representation then adding the forces.




Determine the magnitude of the forces.

Write a vector expression for the resultant couple.

Correct response to preceding frame

$|\bar{F}| = 30\sqrt{5}$ lb

$\bar{C} = 60\bar{j} - 30\bar{k}$ ft-lb

or

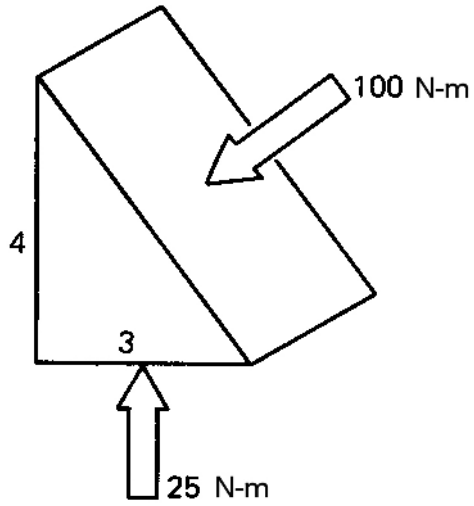
$\bar{C} = 30\sqrt{5}\left[\frac{2\bar{j} - \bar{k}}{\sqrt{5}}\right]$ ft-lb

---

Frame 22-17

**Resultant of Couples**

Find the resultant of the couples shown.




$\bar{R} =$ _____ $\bar{i}$ + _____ $\bar{j}$ N-m

Does a couple have a particular location? ☐ Yes ☐ No

Is it necessary to locate a resultant couple? ☐ Yes ☐ No

Correct response to preceding frame

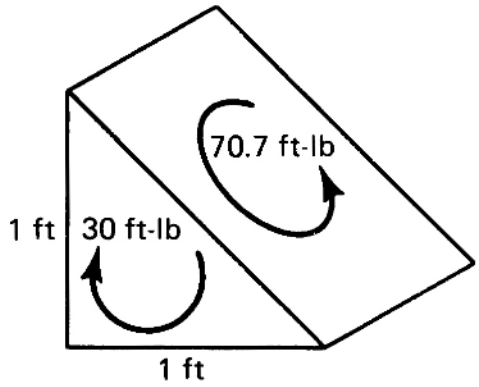
$\bar{R} = -80\bar{i} - 35\bar{j}$ N-m
No
No, couples are free vectors and have no particular location on the body.

---

Frame 22-18

**Resultant of Couples**

Find the resultant of the couples shown.



Correct response to preceding frame

$\overline{R} = 50\overline{i} + 50\overline{j} - 30\overline{k}$ ft-lb

---

Frame 22-19

## Resultants of Couples

The resultant of a system of couples is a (*force, couple*).

Correct response to preceding frame

couple

---

Frame 22-20

## Resultant of Couples

Work Problem 22-2 in your notebook.

Correct response to preceding frame

$\bar{R} = -500\bar{j} + 600\bar{k}$ in-lb

---

Frame 22-21

**Transition**

You have now learned to find the resultant of any concurrent force system, of any coplanar force system, and of any system of couples.

The remainder of this unit will deal with the resultants of non-coplanar force systems.

The next section will teach you to find and locate the resultant of any system of parallel non-coplanar forces. The last section will be devoted to non-concurrent, non-parallel, non-coplanar force systems.

It will take you about 45 minutes to finish this unit and this is a good place to take a break.

When you are ready, go to the next frame.

Correct response to preceding frame

No response

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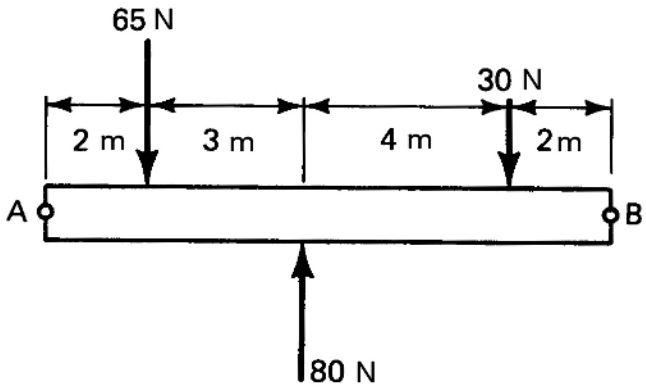
Frame 22-22

**Review**




To determine the resultant of the parallel coplanar force system shown above you would find the magnitude of the resultant force by writing

$\overline{R}$ = _____

and would locate the resultant by writing

_____ = $\Sigma\overline{M}_A$

Correct response to preceding frame

$$\overline{R} = \Sigma \overline{F}$$
$$\overline{a}_A \times \overline{R} = \Sigma \overline{M}_A$$

---

Frame 22-23

**Review**

Is the resultant of a coplanar parallel force system always a force?

☐ Yes ☐ No

If your answer was "No" tell what else it could be. _____

Correct response to preceding frame

No
a couple

---

Frame 22-24

**Resultant of a Parallel Noncoplanar Force System**

To find the resultant force of a parallel noncoplanar force system we write $\overline{R} = \Sigma\overline{F}$.

If $\Sigma\overline{F} = 0$ it implies that the resultant of the force system may be a _____ .

Correct response to preceding frame

couple

---

Frame 22-25

**Resultant of a System of Noncoplanar Parallel Forces**

The resultant of a system of parallel forces in space is always either a force or a couple.

If the resultant is a force it will be _____ to the forces in the system.

If the resultant is a couple, it will be possible to represent it as a pair of forces which

are _____ to the forces of the system.

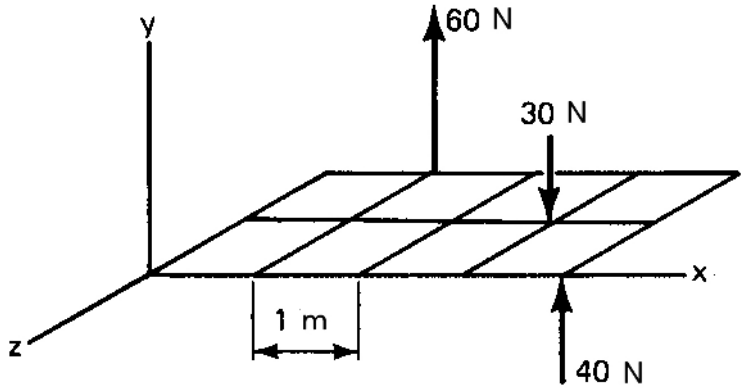
Correct response to preceding frame

parallel
parallel

---

Frame 22-26

**Resultant of a Parallel Noncoplanar Force System**




Write $\overline{R} = \Sigma\overline{F}$ to find the force which is the resultant of the force system shown.

Write $\overline{a}_0 \times \overline{R} = \Sigma\overline{M}_0$ to find the location of the point where the resultant cuts the x-z plane.

Correct response to preceding frame

$\bar{R} = 70\bar{j}$ N

$\bar{a}_o = \frac{13}{7}\bar{i} - \frac{9}{7}\bar{k}$ m

Solution:

$$\bar{R} = \sum \bar{F} = 60\bar{j} + 40\bar{j} - 30\bar{j}$$

$$\bar{a}_o = x_o\bar{i} + z_o\bar{k}$$

$$\left[x_o\bar{i} + z_o\bar{k}\right] \times \left[70\bar{j}\right] = 70x_o\bar{k} - 70z_o\bar{i}$$

$$\sum \bar{M}_o = 4\bar{i} \times 40\bar{j} + \left[\bar{i} - 2\bar{k}\right] \times 60\bar{j} + \left[3\bar{i} - \bar{k}\right] \times -30\bar{j}$$

$$= 160\bar{k} + 60\bar{k} + 120\bar{i} - 90\bar{k} - 30\bar{i} = 130\bar{k} + 90\bar{i}$$

$$70x_o\bar{k} = 70z_o\bar{i} + 130\bar{k} + 90\bar{i}$$

$$x_o = \frac{13}{7} \qquad z_o = -\frac{9}{7}$$

---

Frame 22-27

**Resultant of a System of Parallel Noncoplanar Forces**

Work Problem 22-3 in your notebook.

Correct response to preceding frame

**Solution to problem 22-3**

$\sum \overline{F} = 0$

**Therefore resultant may be a couple.**

$$\sum \overline{M}_o = 4\overline{i} \times -85\overline{j} + \left[2\overline{i} + \overline{k}\right] \times 25\overline{j} + \left[3\overline{i} + 2\overline{k}\right] \times 60\overline{j}$$

$$= -340\overline{k} + 50\overline{k} - 25\overline{i} + 180\overline{k} - 120\overline{i} = -110\overline{k} - 145\overline{i}$$

**Resultant is a couple.**

$\overline{R} = -110\overline{k} - 145\overline{i}$ N-m

---

Frame 22-28

## Resultants

If in finding a resultant of a system you found that $\sum \overline{F} = 0$ and $\sum \overline{M}_A = 0$, what would you conclude? _____

Correct response to preceding frame

That the resultant was zero and the body was in equilibrium

---

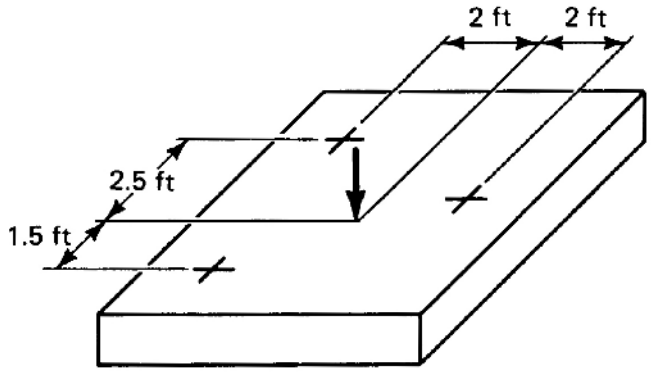
Frame 22-29

**Resultant of a System of Parallel Noncoplanar Forces**

Do Problem 22-4 in your notebook.

Correct response to preceding frame

$\bar{R} = -200\bar{j}$ kips acting as shown.




**Solution:**

$$\bar{R} = \sum \bar{F} = -200\bar{j}$$

Taking 0 at the middle of the plate

$$\sum \bar{M}_o = 2\bar{i} \times -100\bar{j} + (-2\bar{i} - 2\bar{k}) \times -25\bar{j}$$
$$+ (-2\bar{i} + 2\bar{k}) \times -75\bar{j}$$
$$= 100\bar{i} + 0\bar{k}$$
$$\bar{a} \times \bar{R} = [x_o\bar{i} + z_o\bar{k}] \times -200\bar{j}$$
$$= 200\, x_o\bar{k} + 200 z_o\bar{i}$$
$$200x_o\bar{k} + 200z_o\bar{i} = 100\bar{i} + 0\bar{k}$$
$$z_o = \frac{1}{2} \qquad x_o = 0$$

---

Frame 22-30

## Transition

You have now studied the resultants of all the special kinds of force systems.

The only kind of force system left is the general force system in space -- non-concurrent, non-parallel, non-coplanar. Unlike the systems you have studied earlier, the general force system does not have a resultant with a unique form.

To learn to handle this unfortunate situation, turn to the next frame.

Correct response to preceding frame

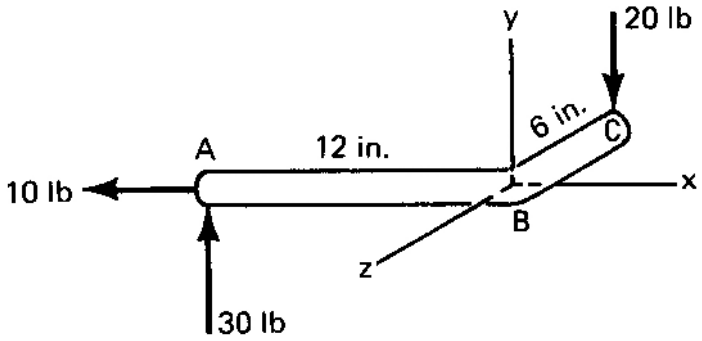
No response

---

Frame 22-31

## Resultant of a General Force System in Space

The resultant of a general force system in space usually consists of a force ***and*** a couple. (0ccasionally, one or the other may be zero.) The resultant force is equal to the sum of the forces and may be placed at any point you choose. The couple is then equal to the sum of the moments about that point.




For the force system shown, what is the resultant force?

$\overline{F}_R$ = _____

If $\overline{F}_R$ acts through A, a couple must be added in order to make the resultant force system equivalent to the original system. The couple must equal the moment about

what point? _____

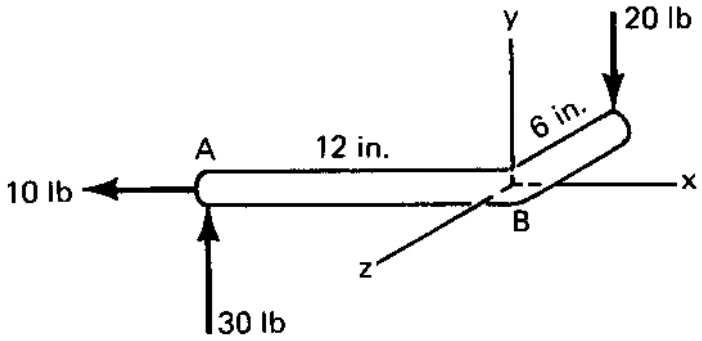
Correct response to preceding frame

$\overline{F}_R = -10\overline{i} + 10\overline{j}$ lb
point A, the point through which $\overline{F}_R$ is made to act

---

Frame 22-32

**Resultant of a General Force System in Space**




1. Find a force system equivalent to the system shown so that the resultant force acts through A.

$\overline{F}_R$ = _____ through A

$\overline{C}_R$ = _____

2. Find a force system equivalent to the system shown so that the resultant force acts through B.

$\overline{F}_R$ = _____ through B

$\overline{C}_R$ = _____

Correct response to preceding frame

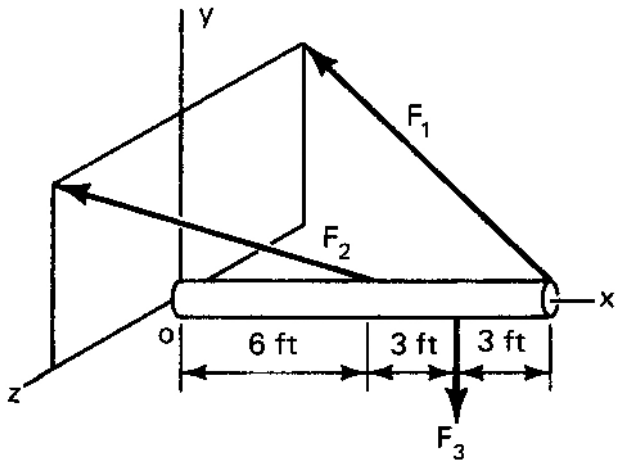
1. $\bar{F}_R = -10\bar{i} + 10\bar{j}$ lb through A
   $\bar{C}_R = -120\bar{i} - 240\bar{k}$ in-lb

2. $\bar{F}_R = -10\bar{i} + 10\bar{j}$ lb through B
   $\bar{C}_R = -120\bar{i} - 360\bar{k}$ in-lb

---

Frame 22-33

**Resultant of a General Force System in Space**




A bar is acted on by the forces shown.

$\bar{F}_1 = -120\bar{i} + 30\bar{j} - 40\bar{k}$ lb
$\bar{F}_2 = -60\bar{i} + 30\bar{j} + 40\bar{k}$ lb
$\bar{F}_3 = -80\bar{j}$ lb

Find an equivalent force system if the resultant force acts through O.

Correct response to preceding frame

$\overline{F}_R = -180\overline{i} - 20\overline{j}$ lb through 0

$\overline{C}_R = -180\overline{k} + 240\overline{j}$ ft-lb

Solution:

$$\overline{C}_R = \sum \overline{M}_o = 9\overline{i} \times -80\overline{j}$$
$$+ 6\overline{i} \times [-60\overline{i} + 30\overline{j} + 40\overline{k}]$$
$$+ 12\overline{i} \times [-120\overline{i} + 30\overline{j} - 40\overline{k}]$$
$$= -720\overline{k} + 180\overline{k} - 240\overline{j} + 360\overline{k}$$
$$+ 480\overline{j}$$
$$= -180\overline{k} + 240\overline{j}$$

$$\overline{F}_R = \sum \overline{F}$$

---

Frame 22-34

**Resultant of a General Force System in Space**

Work Problem 22-5 in your notebook.

Correct response to preceding frame

$\overline{F}_R = -25\overline{j} - 10\overline{i}$ lb through 0
$\overline{C}_R = -150\overline{i} - 250\overline{k}$ in-lb

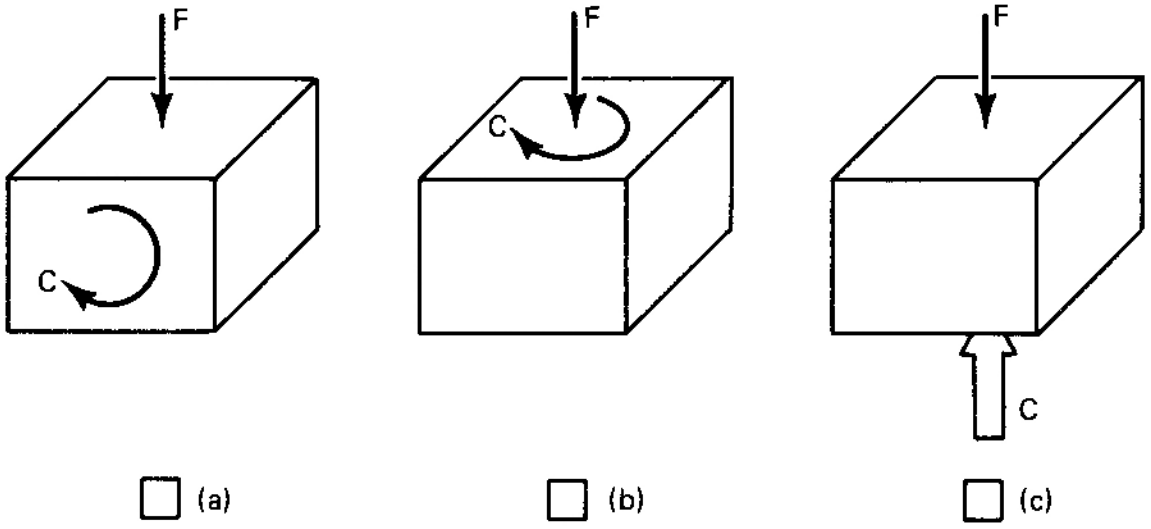
---

Frame 22-35

**Wrenches**

The fact that the resultant of a general force system in space can have so many forms disturbs some people. To be more specific it is possible to reduce any force and couple combination to a couple and a force perpendicular to the plane of the couple (or parallel to the vector representing the couple, if you prefer). This combination is called a wrench (for no apparent reason since it actually acts like a screwdriver).

Which of the following shows a wrench?




☐ (a) ☐ (b) ☐ (c)

Correct response to preceding frame

(b) and (c) represent wrenches

---

Frame 22-36

**Wrenches**

There is only one wrench for a given force system, hence the wrench represents a unique form of the resultant. The force of a given wrench has only one correct location, and the couple of the wrench has only one correct value.

Finding the wrench, is, however, somewhat involved and has little application in elementary mechanics, consequently it will not be discussed further in this program. It is really a concept from more abstract vector mathematics. You or your teacher may wish to work out a method for finding wrenches.

The importance of the wrench is that _____

_____

Correct response to preceding frame

It represents a unique form of the resultant for a general force system.

(Or equivalent response)

---

Frame 22-37

## Summary

Fill out the summary sheet on page 22-5 in your notebook.

Correct response to preceding frame

Read the following summary to check your answer.

---

Frame 22-38

**Summary**

The possible resultants other than zero of force systems can be summarized as follows:

| System | Resultant | Located |
|---|---|---|
| Concurrent | Force = $\sum \overline{F}$ | through point of concurrency |
| Couples | Couple = $\sum \overline{M}$ | anywhere |
| Parallel | Force = $\sum \overline{F}$ Parallel to System Forces<br>or<br>Couple = $\sum \overline{M}$ | by use of $\overline{a}_A \times \overline{R} = \sum \overline{M}_A$<br>anywhere |
| Coplanar | Force = $\sum \overline{F}$<br>or<br>Couple = $\sum \overline{M}$ | by use of $\overline{a}_A \times \overline{R} = \sum \overline{M}_A$ |
| General (Non-Concurrent, Non-Coplanar, Non-Parallel) | Force = $\sum \overline{F}$<br>and<br>Couple = $\sum \overline{M}_A$ | through any arbitrary point A |

If, for any system, $\sum \overline{F} = 0$ and $\sum \overline{M} = 0$, the system is in equilibrium.

This concludes our program on resultants. You may find them a useful tool in simplifying complex force systems. Such a simplification will enable you to more accurately predict the effect of the force system.